Simultaneous Equations

What are they ?

Consider the following:

At the cinema, 2 adult tickets and 3 child tickets cost £17.

Let the cost of the adult ticket = £ x
Let the cost of the child ticket = £ y

We can express this relationship algebraically, in the form of an equation.

$$2x+3y=17$$

However, we cannot solve this equation, there are 2 variables, and any number of solutions.

e.g. The following are all valid solutions

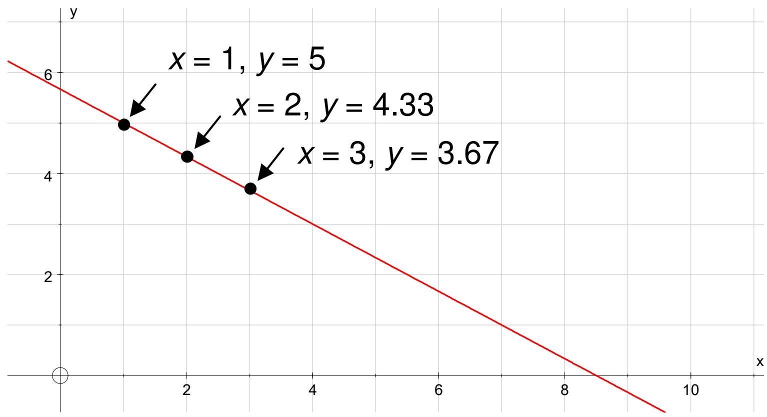
x = 1, and y = 5

x = 2, and y = 4.33

x = 3, and y = 3.67

and so on.

If we were to draw a graph of this relationship, then any point on the line would be a solution.



Since all of those values would satisfy the relationship that:

2 adult tickets and 3 child tickets cost £17

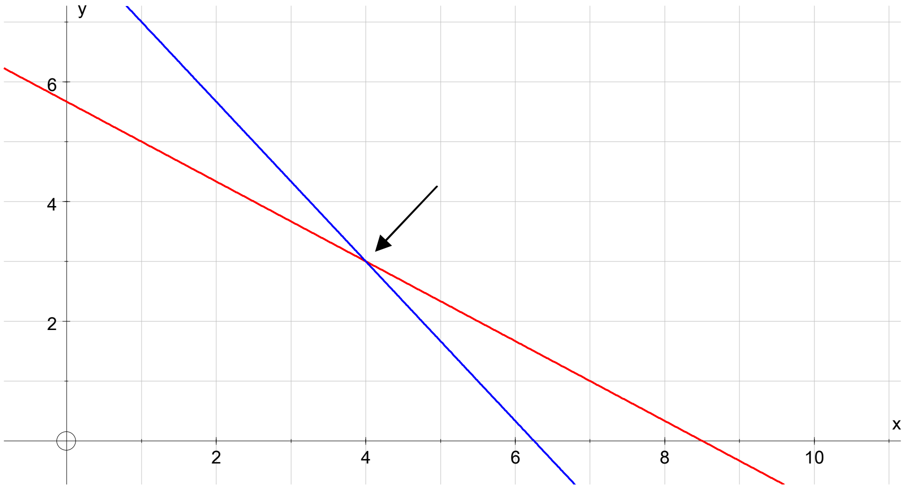
We need more information, to narrow it down to a specific solution.

If we are further told that:

At the same cinema, 4 adult tickets and 3 child tickets cost £25.
then we can form another equation.

$$4x + 3y = 25$$

We can now plot this equation on the same graph as previously.



Both of these relationships have to be true.

2 adult tickets and 3 child tickets cost £17
4 adult tickets and 3 child tickets cost £25

In other words they have to be true at the same time – i.e. simultaneously.

We call the pair of equations that model, or represent this situation

Simultaneous Equations

Looking at the graph, the solution has to be on both graphs.

This can only occur where the lines cross.

So the solution is x = 4, y = 3.

To translate back to the original problem – Adult ticket costs £4 and Child ticket costs £3

Solving simultaneous Equations

We can always draw a graph to solve simultaneous equations, but this is time consuming and may not be accurate.

An algebraic method is preferable.

Method 1. – Substitution

This would be used where the pair of equations are of the form:

$$y = 3x + 5$$
$$y = 5x + 1$$

We note that y is the same in both equations, so equate the right hand sides:

$$5x + 1 = 3x + 5$$

Now solve, as you would a simple equation

$$5x - 3x = 5 - 1$$
$$2x = 4$$
$$x = 2$$

To find y substitute the value of x into either of the original equations (since both are true) e.g. use the first one.

$$y = 3x + 5$$
$$y = 3(2) + 5$$
$$y = 11$$

Hence solution is $x = 2$, $y = 11$

Try these:

1. $y = 3x + 1$
   $y = 6 - 2x$ [Ans. x = 1, y = 4]

2. $y = 2x - 8$
   $y = x + 1$ [Ans. x = 9, y = 10]

3. $y = 2x - 3$
   $x + y = 3$ [Ans. x = 2, y = 1]

   [Hint: replace y with $2x - 3$ in the second equation.]

Method 2 – Elimination

Use this method, when substitution is not convenient, or easy to do.

This method is used for examples like the one in the introduction.

Example:
$$2x + 3y = 17$$
$$4x + 3y = 25$$

Label the equations (1) and (2).

$$2x + 3y = 17 \quad ...(1)$$
$$4x + 3y = 25 \quad ...(2)$$

The aim is to eliminate one of the variables by adding or subtracting the equations.

Here we will eliminate y.

(2) – (1) $$2x = 8$$

(Since 4x – 2x = 2x; 3y – 3y = 0 and 25 – 17 = 8)

and so: $$x = 4$$

Now substitute back in either (1) or (2) – we will choose (1) as the numbers look easier.

$$2x + 3y = 17$$
$$2(4) + 3y = 17$$
$$3y = 17 - 8$$
$$3y = 9$$
$$y = 3$$

Hence our solution is: $x = 4$, $y = 3$ as found from the graph.

Try these:

1. $x + y = 15$, $x - y = 5$ [Ans: x = 10, y = 5]

2. $4s - 3t = 15$, $2s + 3t = 3$ [Ans: s = 3, t = –1]

3. $5m - 2n = 13$, $m - 2n = 1$ [Ans. m = 3, n = 1]

Sometimes, you cannot simply add or subtract and eliminate a variable.

An extended method of elimination

Consider the following example:

$$3x + y = 6 \quad ...(1)$$
$$x - 2y = 2 \quad ...(2)$$

Adding or subtracting will not eliminate a variable.

However, if we were to multiply both sides of equation (1) by 2, then we get:

(1) × 2 $\quad 6x + 2y = 12$

(2) $\quad x - 2y = 2$

Now add, and we get: $\quad 7x = 14$

and so, $\quad x = 2$

Substitute into (2) $\quad x - 2y = 2$

$$2 - 2y = 2$$
$$y = 0$$

Hence solutions are: $x = 2$, $y = 0$

Sometimes we have to multiply both equations, in order to eliminate a variable.

Example:

$$3x + 4y = 20 \quad ...(1)$$
$$4x - 3y = 10 \quad ...(2)$$

This time, we multiply (1) by 3 and (2) by 4 and then add.

(1) × 3 $\quad 9x + 12y = 60$

(2) × 4 $\quad 16x - 12y = 40$

Adding we get: $\quad 25x = 100 \quad$ so, $\quad x = 4$

And substituting into (1) gives: $\quad 3(4) + 4y = 20 \quad$ i.e. $\quad 4y = 12 \quad$ and so $\quad y = 3$

Applications

1 a) 4 peaches and 3 grapefruit cost £1.30
Write down an algebraic equation to illustrate this.

b) 2 peaches and 4 grapefruit cost £1.20.
Write down an algebraic equation to illustrate this.

c) Find the cost of 3 peaches and 2 grapefruit.

[Ans. $4p + 3g = 130$; $2p + 4g = 120$; $g = 22$; $p = 16$; cost for (c) = 92p]

2. Andrew and Doreen each book in at the Sleepwell Lodge.

a) Andrew stays for 3 nights and has breakfast on 2 mornings.
His bill is £145
Write down an algebraic equation to illustrate this.

b) Doreen stays for 5 nights and has breakfast on 3 mornings.
Her bill is £240.
Write down an equation to illustrate this.

c) Find the cost of one breakfast.

[Ans. $3n + 2b = 145$; $5n + 3b = 240$; $n = 45$; $b = 5$; One breakfast = £5]

3. On a ferry crossing, 3 caravans and two cars cost £205,
and 2 caravans and 3 cars cost £195.

Find the cost for a car and for a caravan.

[Ans. $3v + 2c = 205$; $2v + 3c = 195$; $v = 45$; $c = 35$; car cost: £35; caravan cost: £45]

4. An adults train fare is £2 more than a child's. The adult's fare is twice the child's.
Find the cost of each fare.

[Hint: Let adult fare = £x and child fare = £y: So: $x = y + 2$ and $x = 2y$]
[Ans: $x = 4$; $y = 2$; adult fare = £4, child fare = £2]