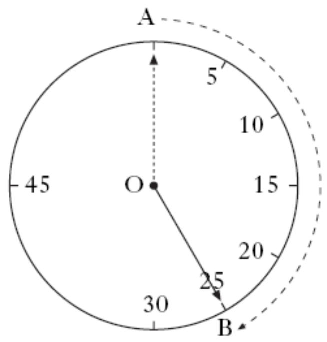
Problem Solving Questions 3

1. Contestants in a quiz have 25 seconds to answer a question.
This time is indicated on the clock.
The tip of the clock hand moves through the arc AB as shown.



(*a*) Calculate the size of angle AOB. **1 KU**

There are 12 lots of 5 minutes in 1 hour, so 5 minutes corresponds to 360 ÷ 12 = 30°

Hence angle AOB = 5 × 30 = 150°

(*b*) The length of arc AB is 120 centimetres.

Calculate the length of the clock hand. **4 RE**

Length of arc AB = 120 = $\frac{150}{360} \times \pi \times D$

Rearranging we get: $120 \times 360 = 150\pi D$

And so: $D = \frac{120\times360}{150\pi}$ D = 91.67 cm

Length of clock hand = radius = 91.67 ÷ 2 = 45.8 cm (3 sig. fig)

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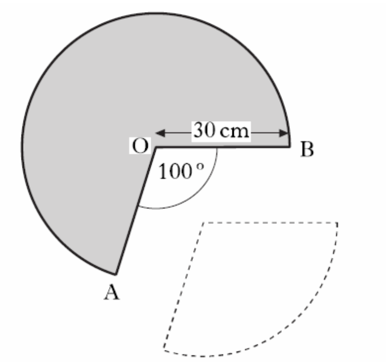
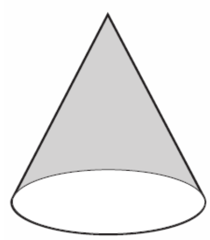
2. A cone is formed from a paper circle with a sector removed as shown.
The radius of the paper circle is 30 cm.
Angle AOB is 100°.





a) Calculate the area of paper used to make the cone. **3 KU**

Angle of major sector = 360 – 100 = 260°.

Area of sector = $\frac{260}{360} \times \pi \times 30^2$ Area of paper = 2042 cm^2.

b) Calculate the circumference of the base of the cone. **3 RE**

Circumference = arc length original part of circle = $\frac{260}{360} \times \pi \times 60$ = 136 cm

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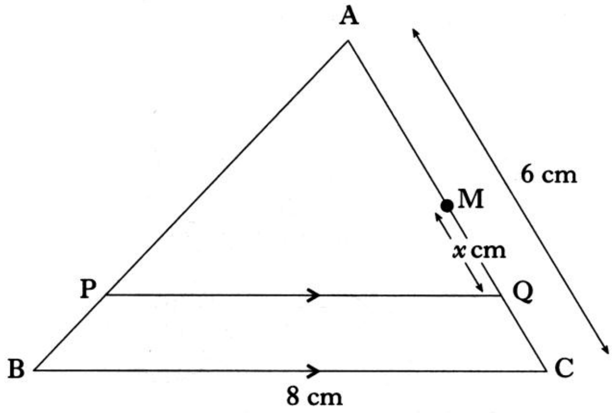
3. In triangle ABC

BC = 8 centimetres

AC = 6 centimetres

PQ is parallel to BC



M is the mid-point of AC
Q lies on AC, x centimetres from M,
as shown on the diagram.

(a) Write down an expression for the length of AQ. **1 RE**

AM = 3 cm So AQ = 3 + x

(b) Show that $PQ = \left(4 + \frac{4}{3}x\right)$ **3 RE**

Use similar triangles

$\frac{PQ}{8} = \frac{AQ}{AC}$. Substituting gives: $\frac{PQ}{8} = \frac{3+x}{6}$

cross multiply $PQ = \frac{8(3+x)}{6}$ now break bracket: $PQ = \frac{24+8x}{6}$

Split into two fractions: $PQ = \frac{24}{6} + \frac{8x}{6}$ and simplify (cancel): $PQ = 4 + \frac{4x}{3}$

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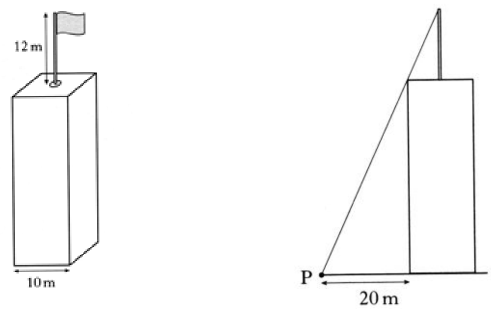
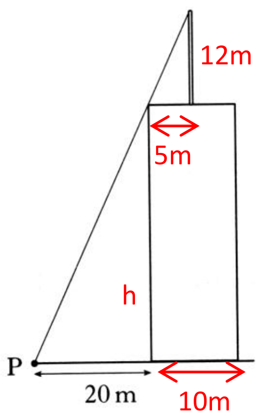
4. A vertical flagpole 12 metres high stands at the centre of the roof of a tower.
The tower is cuboid shaped with a square base of side 10 metres.



At a point P on the ground, 20 metres from the base of the tower, the top of the flagpole is just visible, as shown.
Calculate the height of the tower. **4 RE**



Mark all information onto the diagram,
Use similar triangles:

$\frac{h}{12} = \frac{20}{5}$ Now cross multiply: $h = \frac{12 \times 20}{5}$

Height of tower = 48 metres.

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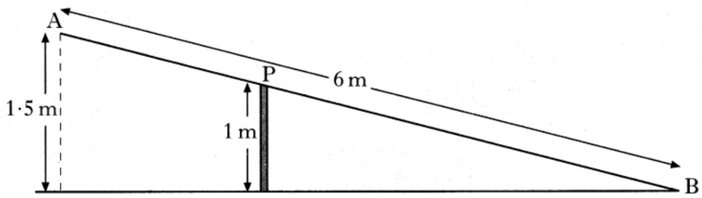
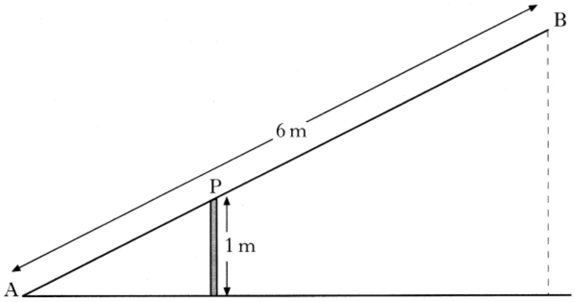
5. A metal beam, AB, is 6 metres long.
It is hinged at the top, P, of a vertical post 1 metre high.
When B touches the ground, A is 1.5 metres above the ground, as shown
In Figure 1.

Figure 1



When A comes down to the ground, B rises, as shown in Figure 2.



By calculating the length of AP, or otherwise, find the height of B above the ground.

Do not use a scale drawing. **5 RE**

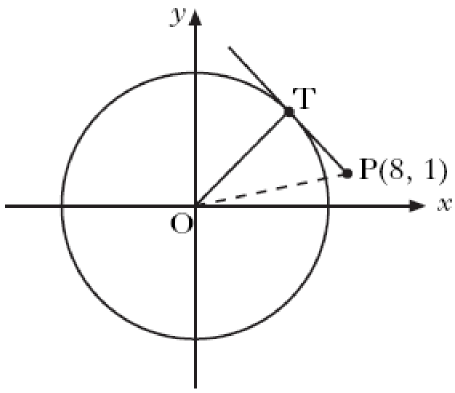
In fig 1. Use similar triangles to calculate PB: $\frac{PB}{6} = \frac{1}{1.5}$ cross multiply $PB = \frac{6}{1.5} = 4$

In fig 2: Note that since PB = 4 metres, then AP = 2 metres.

Now use similar triangles again in fig 2: $\frac{h}{1} = \frac{6}{2}$ hence h = 3

so height of B above the ground is 3 metres.

6. A circle, centre the origin, is shown.
P is the point (8, 1).



(a) Calculate the length of OP. **2 RE**

Using Pythagoras:

$OP^2 = 1^2 + 8^2$

$OP^2 = 65$

$OP = \sqrt{65} = 8.06$

The diagram also shows a tangent from P which touches the circle at T.
The radius of the circle is 5 units.

(b) Calculate the length of PT. **2 RE**

Since PT is a tangent, then angle OTP = 90°

OT = radius = 5. OP = 8.06 (found in part (a))

Using Pythagoras again, being careful to identify hypotenuse:

$OP^2 = OT^2 + PT^2$ Now substitute values in (Note that we can use $OP^2 = 65$)

$65 = 5^2 + PT^2$ so, $PT^2 = 65 - 25 = 40$ Hence $PT = \sqrt{40} = 6.324\ldots$

Length of PT = 6.3 units (2 sig. fig.)

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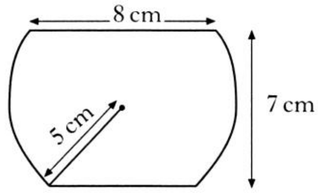
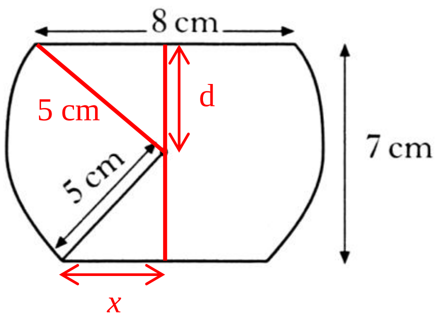
7. A badge is made from a circle of radius 5 centimetres.
Segments are taken off the top and the bottom of the circle as shown.
The straight edges are parallel.



The badge measures 7 centimetres from the top to the bottom.
The top is 8 centimetres wide.
Calculate the width of the base. **5 RE**



Using Pythagoras in top triangle:

$5^2 = 4^2 + d^2 \rightarrow d^2 = 5^2 - 4^2 = 25 - 16 = 9$

Hence d = 3 cm,

So height of lower triangle is 7 – 3 = 4 cm

Using Pythagoras in lower triangle:

$5^2 = 4^2 + x^2 \rightarrow x^2 = 5^2 - 4^2 = 25 - 16 = 9$

Hence $x = 3$ and so width of base = 3 × 2 = 6 cm

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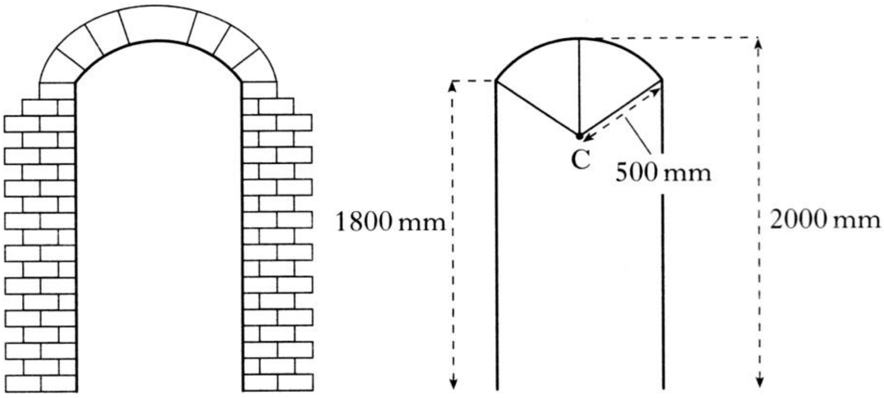
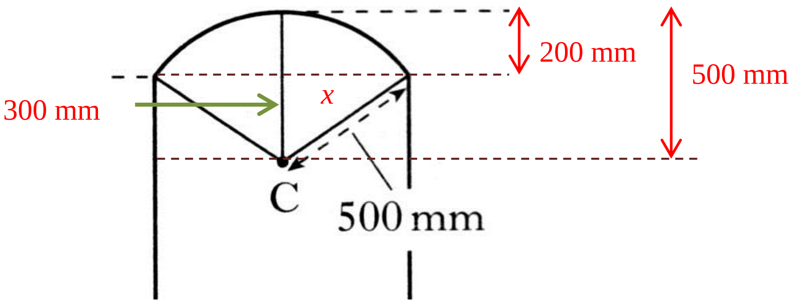
8. The curved part of a doorway is an arc of a circle with radius 500 millimetres and centre C.

The height of the doorway to the top of the arc is 2000 millimetres.

The vertical edge of the doorway is 1800 millimetres.



Calculate the width of the doorway. **5 RE**



Using Pythagoras

$500^2 = 300^2 + x^2$

Re-arranging

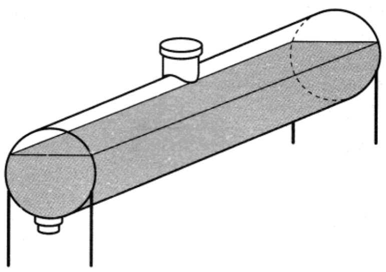
$x^2 = 500^2 - 300^2$

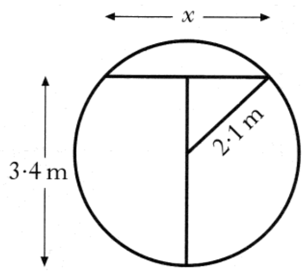
$x^2 = 160,000$

$x = 400$ mm

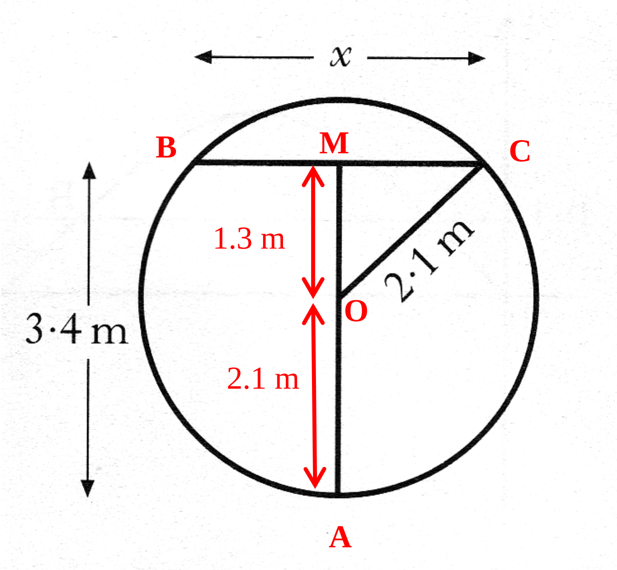
Hence width of door is 400 × 2 = 800 mm

9. An oil tank has a circular cross section of radius 2.1 metres.
It is filled to a depth of 3.4 metres.





(a) Calculate x, the width in metres of the oil surface. **3 KU**



Label the diagram as shown.

OA is a radius and so is 2.1 m

Hence OM = AM – 2.1 = 3.4 – 2.1 = 1.3 m

Using Pythagoras in triangle OMC.

$2.1^2 = 1.3^2 + MC^2$

Re-arranging,

$MC^2 = 2.1^2 - 1.3^2$

$MC^2 = 2.72$

MC = 1.649…

And so $x = 2 \times MC = 2 \times 1.649\ldots = 3.298\ldots$

Hence the width of the oil surface = 3.3 m (2 sig figs)

(b) What other depth of oil would give the same surface width. **1 RE**

By symmetry, rotate the diagram through 180°,

Diameter of tank is 4.2 metres.

And the other depth of oil = 4.2 – 3.4 = 0.8 metres.

10. A sheep shelter is part of a cylinder as shown in Figure 1.

It is 6 metres wide and 2 metres high.

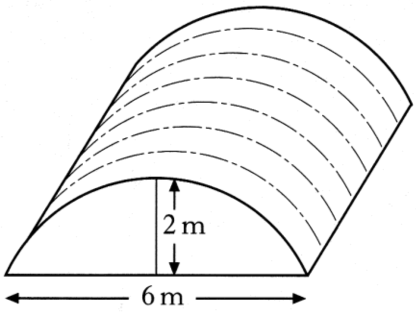


Figure 1

The cross-section of the shelter is a segment of a circle with centre O, as shown in Figure 2.

OB is the radius of the circle.

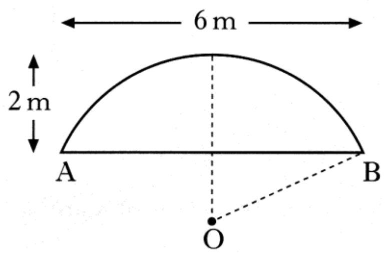


Figure 2

Calculate the length of OB. **4 RE**

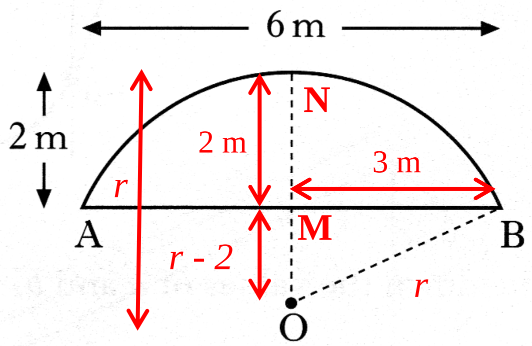


Figure 2

Label the diagram as shown.

OB is a radius. MB = 6 ÷ 2 = 3 metres

ON = r, MN = 2, so OM = $r - 2$ metres.

Using Pythagoras in triangle OMB.

$r^2 = 3^2 + (r - 2)^2$

$r^2 = 9 + r^2 - 4r + 4$

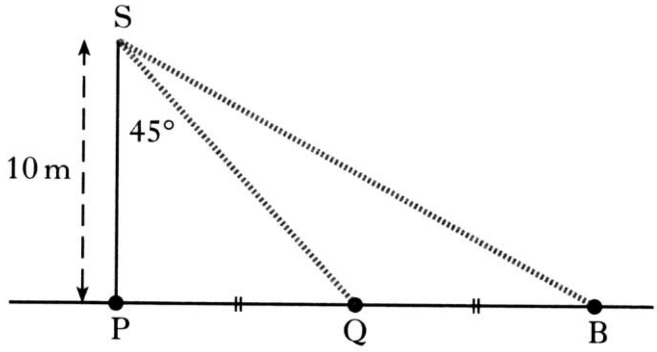
$\not{r^2} = 13 + \not{r^2} - 4r$

$4r = 13$ and so $r = 13 \div 4 = 3.33333\ldots$

Length OB = 3.33 metres.

11. The diagram below shows a spotlight at point S, mounted 10 metres directly above a point P at the front edge of a stage.

The spotlight swings 45° from the vertical to illuminate another point Q, also at the front edge of the stage.



Through how many more degrees must the spotlight swing to illuminate a point B, where Q is the mid–point of PB ? **5 RE**

We need to find the angle QSB. Note that Triangle SPQ is right angled.

Angle PQS is also 45°, so this triangle is isosceles. Thus PQ = SP = 10 metres.

Since PQ = QB then PB = 20 metres.

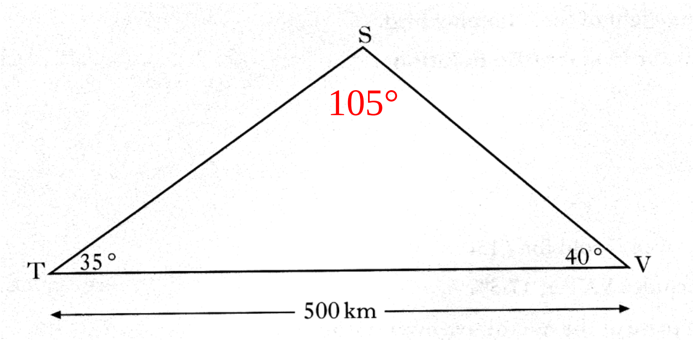
Use SOH-CAH-TOA in triangle SPB.

$\tan PSB = \frac{20}{10} = 2$ so $\angle PSB = \tan^{-1}(2)$ $\angle PSB = 63.4°$

Hence: $\angle QSB = 63.4° - 45° = 18.4°$

Spotlight must swing a further 18.4° to illuminate point B.

12. A TV signal is sent from a transmitter T, via a satellite S, to a village V, as shown in the diagram. The village is 500 kilometres from the transmitter.



The signal is sent out at an angle of 35° and is received in the village at an angle of 40°.

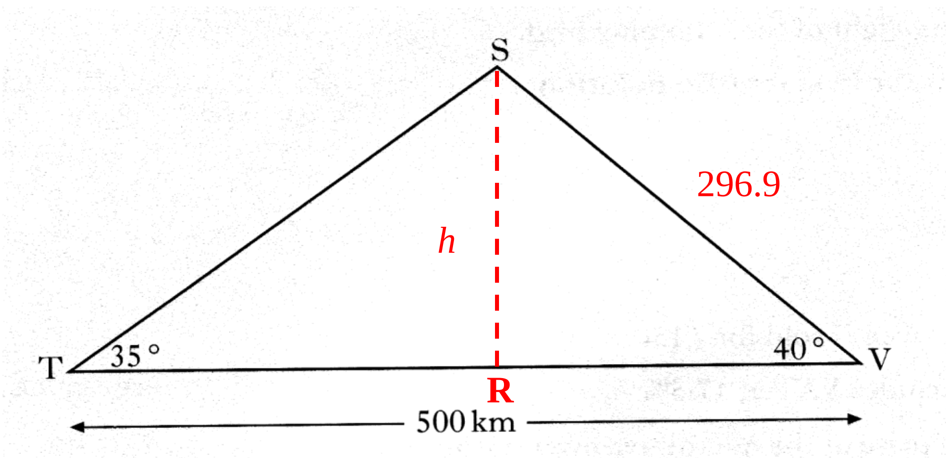
Calculate the height of the satellite above the ground. **5 RE**

First find another side in the triangle – either ST or SV.

Note that angle TSV = 105°

Use the sine rule: $\frac{SV}{\sin 35°} = \frac{500}{\sin 105°}$

Re-arranging: $SV = \frac{500 \times \sin 35°}{\sin 105°} = 296.9\ km$



Using SOH-CAH-TOA in triangle SRV: $\sin 40° = \frac{h}{296.9}$

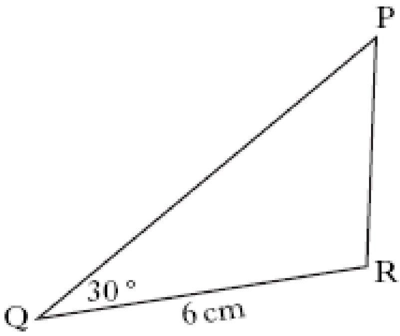
So, $h = 296.9 \times \sin 40° = 190.84.....$ Thus height of satellite is 191 km.

13. In triangle PQR

QR = 6 centimetres

Angle PQR = 30°

Area of triangle PQR = 15 square centimetres.



Calculate the length of PQ. **3 RE**

Use: Area of triangle = ½ a b sin C

So: 15 = ½ × PQ × 6 × sin 30°

Simplify and re-arrange. Note that sin 30° = ½ and we get: 15 = ½ × 6 × ½ × PQ

½ × ½ = ¼ and ¼ × 6 = 1.5 so 15 = 1.5 PQ and PQ = 15/1.5 = 10.

Length PQ = 10 cm (We can check this: Area = ½ × 10 × 6 × sin 30° → 30 × ½ = 15)

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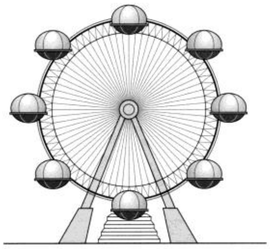
14. Emma goes on the "Big Eye".

Her height, h metres, above the ground is given by the formula

$$h = -31\cos t° + 33$$

where t is the number of seconds after the start.



(a) Calculate Emma's height above the ground 20 seconds after the start. **2 KU**

Put $t = 20$ in formula: $h = -31\cos 20° + 33 \rightarrow h = 3.9\,metres$

(b) When will Emma first reach a height of 60 metres above the ground ? **3 RE**

Put h = 60 in the formula and find t.

$$60 = -31\cos t° + 33$$
$$31\cos t° = -27$$
$$\cos t° = -\frac{27}{31} \qquad ***$$

Solve equation: $t = \cos^{-1}\left(\frac{27}{31}\right) \rightarrow$ *acute* $t = 29.4°$

However, note that cosine is negative ***

So using ASTC, we find first solution is in 2^{nd} quadrant and is 180 – 29.4 = 150.6°

Degrees are representing seconds.

Emma will first reach a height of 60 metres above ground after 151 seconds.

(c) When will she next be at a height of 60 metres above the ground. **1 RE**

The next time she will be at 60 metres will be the next solution in the 3^{rd} quadrant

i.e. 180 + 29.4 = 209.4° which corresponds to 209 seconds.

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15. Given $f(x) = 4\sqrt{x} + \sqrt{2}$

(a) Find the value of $f(72)$ as a surd in its simplest form. **3 KU**

$$f(x) = 4\sqrt{x} + \sqrt{2}$$
$$f(72) = 4\sqrt{72} + \sqrt{2}$$
$$f(72) = 4\sqrt{36 \times 2} + \sqrt{2}$$
$$f(72) = 4 \times 6\sqrt{2} + \sqrt{2}$$
$$f(72) = 24\sqrt{2} + \sqrt{2} = 25\sqrt{2}$$

(b) Find the value of t, given that $f(t) = 3\sqrt{2}$. **3 RE**

$$f(x) = 4\sqrt{x} + \sqrt{2}$$
$$f(t) = 4\sqrt{t} + \sqrt{2}$$
$$3\sqrt{2} = 4\sqrt{t} + \sqrt{2}$$
$$2\sqrt{2} = 4\sqrt{t}$$
$$\sqrt{2} = 2\sqrt{t}$$

square each side

$$2 = 4t \quad \rightarrow \quad t = \frac{1}{2}$$

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16. The sum S_n of the first n terms of a sequence, is given by the formula

$$S_n = 3^n - 1$$

(a) Find the sum of the first 2 terms. **1 RE**

$$S_2 = 3^2 - 1 = 9 - 1 = 8$$

(b) When $S_n = 80$, calculate the value of n. **2 RE**

$$S_n = 3^n - 1$$
$$80 = 3^n - 1$$
$$81 = 3^n$$

We are looking for a power of 3 that equals 81.

$3^2 = 9$; $3^3 = 27$; $3^4 = 81$;

So, $n = 4$

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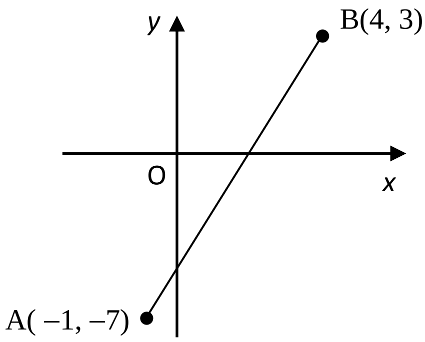
17. In the diagram, A is the point (–1, 7) and B is the point (4, 3).



(a) Find the gradient of the line AB. **1 KU**

Gradient = $m = \dfrac{rise}{run}\left(or \ \dfrac{y_2 - y_1}{x_2 - x_1} \right) = \dfrac{3-(-7)}{4-(-1)} = \dfrac{10}{5} = 2$

(b) AB cuts the *y*-axis at the point (0, –5).
Write down the equation of the line AB **1 KU**

The y-intercept is c = -5 and since m = 2 then using y = mx + c

We get: $y = 2x - 5$

(c) The point (3*k*, *k*) lies on AB
Find the value of k. **2 RE**

Since point (3*k*, *k*) lies on the line, then it satisfies the equation of the line.

Thus: $k = 2(3k) - 5$

Simplifying: $k = 6k - 5 \quad \rightarrow \quad 5 = 5k \quad \rightarrow \quad k = 1$

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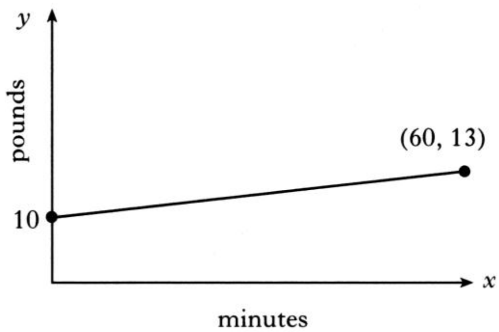
18. The monthly bill for a mobile phone is made up of a fixed rental plus call charges. Call charges vary as the time used.

The relationship between the monthly bill, *y* (pounds), and the time used, *x* (minutes) is represented in the graph below.



(a) Write down the fixed rental. **1 RE**

Fixed rental is when 0 minutes are used i.e. (0, 10). Fixed rental is £10 per month.

(b) Find the call charge per minute. **3 RE**

This is the gradient. i.e. how many pounds per minute.

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{13-10}{60-0} = \dfrac{3}{60}$ Note that this is in £

To change to pence, multiply by 100: $\dfrac{3}{60} \times \dfrac{100}{1} = \dfrac{300}{60} = 5\,p$ per minute

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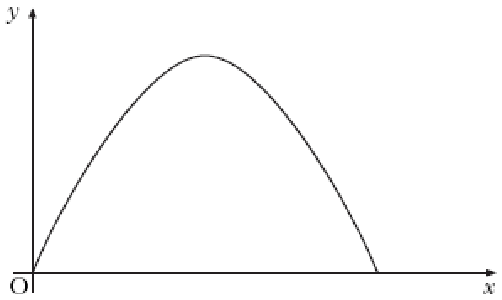
19. The profit made by a publishing company of a magazine is calculated by the formula

$$y = 4x(140 - x),$$

where y is the profit (in pounds) and x is the selling price (in pence) of the magazine.
The graph below represents the profit y against the selling price x.



Find the maximum profit the company can make from the sale of the magazine. **4 RE**

Solve the equation: $4x(140 - x) = 0$ to find where graph cuts x axis.

Solution is: $x = 0$ and $x = 140$

Maximum profit (y) is at turning point which lies mid-way between roots.
i.e. when $x = 70$

Now use equation to calculate maximum profit (when $x = 70$):

$$y = 4x(140 - x)$$

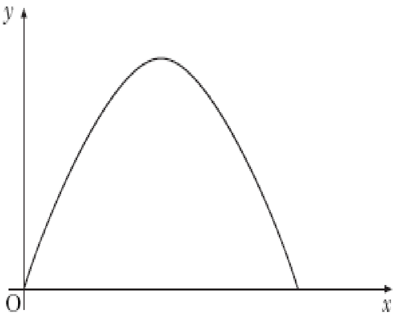
$$y = 4(70)(140 - 70)$$

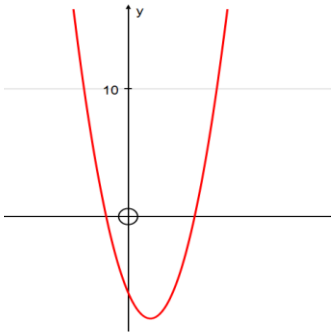
$$y = 40 \times 70 \times 70 \quad = \quad £196,000$$

20. The diagram shows part of the graph of a quadratic function, with equation of the form

$$y = k(x-a)(x-b)$$

The graph cuts the y-axis at (0, –6) and the x-axis at (–1, 0) and (3, 0)





Note: It appears that the incorrect diagram was shown with this question. The correct diagram is shown above on the right.

(a) Write down the values of a and b. **2 KU**

a and b are where it cuts the x-axis: so a = -1 and b = 3

(b) Calculate the value of k. **2 KU**

Putting in values of a and b. $y = k(x+1)(x-3)$

The graph cuts the y-axis when x = 0 and y = -6, so put this point in the equation.

$$-6 = k(0+1)(0-3) \rightarrow -6 = k(1)(-3)$$
$$-6 = -3k \rightarrow k = 2$$

(c) Find the coordinates of the minimum turning point of the function **2 RE**

Turning point is mid-way between the roots of x = -1 and x = 3

so x coordinate of turning point is 1

To find the y-coordinate of turning point, put x = 1 into the equation

$$y = 2(x+1)(x-3) \rightarrow y = 2(1+1)(1-3) \rightarrow y = 2(2)(-2) \rightarrow y = -8$$

Coordinates of minimum turning point are : (1, -8)

21. A rectangular wall vent is 30 centimetres long and 20 centimetres wide.

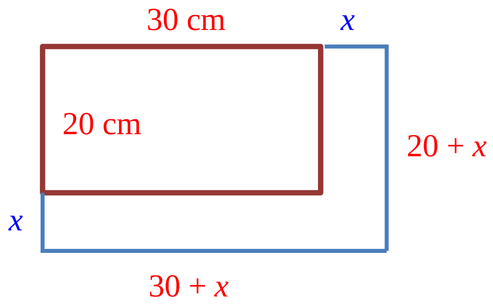
It is to be enlarged by increasing **both** the length and the width by x centimetres.

(a) Write down the length of the new vent. **1 RE**

New length = 30 + x cm

(b) Show that the Area, A, square centimetres, of the new vent is given by

$$A = x^2 + 50x + 600$$ **2 RE**



Area of new vent:

$$A = (30 + x)(20 + x)$$
$$A = 600 + 30x + 20x + x^2$$
$$A = x^2 + 50x + 600$$

(c) The area of the new vent must be at least 40% more than the original area.

Find the minimum dimensions to the nearest centimetre, of the new vent. **5 RE**

Area of old vent = 30 × 20 = 600 cm^2

Area of new vent to be at least 40% more. i.e. 600 × 1.4 = 840 cm^2

Solve equation: $A = x^2 + 50x + 600$ where A = 840

$$x^2 + 50x + 600 = 840$$
$$x^2 + 50x - 240 = 0$$

Use quadratic formula with a = 1, b = 50, c = -240.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-50 \pm \sqrt{(50)^2 - 4(1)(-240)}}{2(1)}$$
$$x = \frac{-50 \pm \sqrt{2500 + 960}}{2}$$
$$x = \frac{-50 + \sqrt{3460}}{2} \quad or \quad x = \frac{-50 - \sqrt{3460}}{2}$$
$$x = 4.41\ cm \quad or \quad x = -54.41\ cm \quad (DISCARD)$$

x needs to be 4.41 cm as negative value is not possible here.

Minimum dimensions are 35 cm by 25 cm

22. The number of diagonals, d, in a polygon with n sides is given by the formula:

$$d = \frac{n(n-3)}{2}$$

A polygon has 20 diagonals

How many sides does it have? **4 RE**

Put $d = 20$ into formula and solve for n $\quad 20 = \frac{n(n-3)}{2}$

Multiply both sides by 2: $\quad 40 = n(n-3)$

Break the brackets: $\quad 40 = n^2 - 3n$

Re-arrange: $\quad n^2 - 3n - 40 = 0$

Factorise: $\quad (n+5)(n-8) = 0$ Hence: n = -5 or n = 8 Discard n = -5

Polygon has 8 sides.

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23. A number pattern is given below.

1st term: $2^2 - 0^2$

2nd term: $3^2 - 1^2$

3rd term: $4^2 - 2^2$

(a) Write down a similar expression for the 4th term. **1 RE**

4^{th} term: $\quad 5^2 - 3^2$

(b) Hence or otherwise find the nth term in its simplest form. **3 RE**

n^{th} term: $\quad (n+1)^2 - (n-1)^2$

$n^2 + 2n + 1 - (n^2 - 2n + 1)$

Simplify: $\quad n^2 + 2n + 1 - n^2 + 2n - 1$

$4n$

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24. (a) One session at the Leisure Centre costs £3.

Write down an algebraic expression for the cost of x sessions **1 RE**

£ $3x$

(b) The Leisure Centre also offers a monthly card costing £20.
The first 6 sessions are then free, with each additional session costing £2.

(i) Find the total cost of a monthly card and 15 sessions. **1 KU**

£ $20 + 2 \times (15 - 6)$

(ii) Write down an algebraic expression for the total cost of a monthly card and x sessions where x is greater than 6. **2 RE**

£ $20 + 2 \times (x - 6)$

(c) Find the minimum number of sessions required for the monthly card to be the cheaper option.

Show all working. **3 RE**

We require $20 + 2 \times (x - 6) < 3x$

$20 + 2x - 12 < 3x$

$8 + 2x < 3x$

$8 < x$

$x > 8$ Minimum number of sessions is 9.

25. A new fraction is obtained by adding x to the numerator and denominator of the fraction $\frac{17}{24}$.

This new fraction is equivalent to $\frac{2}{3}$.

Calculate the value of x. **3 RE**

$$\frac{17+x}{24+x} = \frac{2}{3}$$

Cross multiply: $3(17+x) = 2(24+x)$

Solve equation: $51 + 3x = 48 + 2x$

$3x - 2x = 48 - 51$

$x = -3$

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26. To hire a car costs £25 per day plus a mileage charge.

The first 200 miles are free with each additional mile charged at 12 pence.

> **CAR HIRE**
>
> **£25 per day**
>
> • **first 200** miles free
> • each additional mile only 12p

(a) Calculate the cost of hiring a car for 4 days when the mileage is 640 miles. **1 KU**

$£\ 25 \times 4 + 0.12\ (640 - 200)$

(b) A car is hired for d days and the mileage is m miles where $m > 200$. Write down a formula for the cost £C of hiring the car. **3 RE**

$C = 25 \times d + 0.12\ (m - 200)$

$C = 25d + 0.12\ (m - 200)$

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27. The n^{th} term, T_n of the sequence 1, 3, 6, 10 ….. is given by the formula:

$T_n = \frac{1}{2}n(n+1)$ 1st term $T_1 = \frac{1}{2} \times 1(1+1) = 1$

2nd term $T_2 = \frac{1}{2} \times 2(2+1) = 3$

3rd term $T_3 = \frac{1}{2} \times 3(3+1) = 6$

(a) Calculate the 20th term, T_{20}. **1 KU**

$$T_{20} = \frac{1}{2} \times 20(20+1) = 210$$

(b) Show that $T_{n+1} = \frac{1}{2}(n^2 + 3n + 2)$ **2 RE**

$$T_{n+1} = \frac{1}{2} \times (n+1)((n+1)+1)$$
$$T_{n+1} = \frac{1}{2} \times (n+1)(n+2)$$
$$T_{n+1} = \frac{1}{2} \times (n^2 + 2n + n + 2)$$
$$T_{n+1} = \frac{1}{2} \times (n^2 + 3n + 2)$$

(c) Show that $T_n + T_{n+1}$ is a square number **2 RE**

$$T_n = \frac{1}{2} \times n(n+1)$$

And $T_{n+1} = \frac{1}{2} \times (n^2 + 3n + 2)$

Adding we get:

$$\begin{aligned} T_n + T_{n+1} &= \frac{1}{2} \times n(n+1) + \frac{1}{2} \times (n^2 + 3n + 2) \\ &= \frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)(n+1) \\ &= (n+1)^2 \end{aligned}$$

28. A number pattern is shown below.

$$1^3 = \frac{1^2 \times 2^2}{4}$$

$$1^3 + 2^3 = \frac{2^2 \times 3^2}{4}$$

$$1^3 + 2^3 + 3^3 = \frac{3^2 \times 4^2}{4}$$

(a) Write down a similar expression for: $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ **1 RE**

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \frac{5^2 \times 6^2}{4}$$

(b) Write down a similar expression for: $1^3 + 2^3 + 3^3 \text{ ...... } + n^3$ **2 RE**

$$1^3 + 2^3 + 3^3 + \text{.......} + n^3 = \frac{n^2 \times (n+1)^2}{4}$$

(c) Hence **evaluate**: $1^3 + 2^3 + 3^3 \text{ ...... } + 9^3$ **2 RE**

Put $n = 9$ in the above formula:

$$1^3 + 2^3 + 3^3 + \text{.......} + 9^3 = \frac{9^2 \times (9+1)^2}{4}$$

$$= \frac{81 \times 100}{4} = \frac{8100}{4} = 2025$$